Network Analysis II: Strategic Perspectives

PS690 Computational Methods in Social Science

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Overview

1. One-Sided Links
2. Two-Sided Links
3. Application: Wars

One-Sided Links

- We start from which individuals can unilaterally decide to form links with others.
- This approach gives rise to a noncooperative game that can be solved using the concept of the Nash equilibrium.
- Consider a set of players $N = \{1, 2, ..., n\}$.
- The strategy of player i is a row vector $s_i = (s_{i,1}, ..., s_{i,i-1}, s_{i,i+1}, ..., s_{i,n})$ where $s_{i,j} \in \{0, 1\}$ denoting the link to j.
- The strategy profile is $s = (s_1, ..., s_n)$.
- Given s, let $\prod_i(s)$ be the payoff of player i.
- The Nash equilibrium is $s^* = (s_1^*, ..., s_n^*)$ such that for every player $i \in N$, $\prod_i(s_i^*, s_{-i}^*) \geq \prod_i(s_i, s_{-i}^*)$ for every s_i.
- A Nash equilibrium is strict if all players choose a strict best response, i.e., the inequalities defining the equilibrium are strict for every player.

One-Sided Links

- There is an equivalence between a strategy profile and a directed network.
- We shall say that $N_i^d = \{j \in N | g_{ij} = 1\}$ is the set of players with whom player i forms a link (d for directed).
- Note that $\eta_i^d(g) = |N_i^d|$is the out-degree of player i.
- $N_{-i}^d = \{j \in N | g_{ji} = 1\}$ is the set of players who form a link with player i.
- Note that $\eta_{-i}^d(g) = |N_{-i}^d|$ is the in-degree of player i.
- Let $\mathcal{N}_i(g) = \{k | i \to^g k\}$ be the set of individuals accessed through a directed path by i.
- The payoff in network g is $\prod(g) = (\prod_1(g), ..., \prod_n(g))$
- Follow the convention that a player accesses themselves, so the total number of players accessed by player i in network g is $n_i(g) = |\mathcal{N}_i(g)| + 1$.

One-Sided Links

- In the study of network formation, an important concern will be the relation between equilibrium/stable networks and socially desirable networks.
- Two aspects of social desirability will be touched upon: efficiency and equity.
- There are two notions of efficiency:
    1. Pareto efficiency: g is Pareto efficient if there is no other network g' which Pareto-dominates g: $\prod_i(g') \geq \prod_i(g)$ for all i and there is a player j such that $\prod_j(g') > \prod_j(g)$.
    2. Aggregate efficiency: Network g is said to be efficient if aggregate welfare $W(g) = \sum_{i \in N} \prod_i(g) > W(g')$ for all g'.

One-Sided Links

- For equity, standard measures of inequality include the range, variance, and Gini coefficient.
- We will sometimes also consider the ratio of maximum versus minimum (or the ratio of maximum/median payoffs).
- The range of the payoffs:

$$R(g) = \max_{i \in N} \prod_i (g) - \min_{j \in N} \prod_j (g)$$

- The variance of payoffs:

$$Var(g) = \frac{\sum_i^n [\prod_i(g) - \overline{\prod}_i(g)]^2}{n}$$

One-Way Flow Model

- Consider the benefit of player i depends on $n_i(g)$ and the cost depends on $\eta_i^d(g)$ (maintain links need cost).
- Let $\phi(x, y)$ be strictly increasing in x and strictly decreasing in y.
- Therefor, the payoff function can be represented by $\prod_i(g) = \phi(n_i(g), \eta_i^d(g))$.
- The common linear payoff function $\prod_i(g) = n_i(g) - \eta_i^d(g)k$ for $k > 0$ is a simple example.
- What is the architecture of networks that arise?
- We need to examine the Nash equilibria of the game.

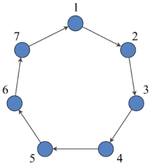
One-Way Flow Model

- The first observation: in equilibrium, either no one forms any links and the network is empty or every individual accesses everyone else, and the network is connected.
- Suppose not. Consider i has paths to **the most** players and i does not observe everyone.
- Then, there must be a player j who is not observed by i and who does not observe i (otherwise, j would access more players than i).
- We argue that j can earn a strictly higher payoff by forming a single link with i; therefor a profitable deviation.
- Suppose that j has formed links that include a link with k. By deleting all their current links and forming a single link with i, they will access strictly more players than i, since they have the additional benefit of observing i.
- Since j was observing weakly fewer individuals than i in their original strategy, and they are forming weakly fewer links in this deviation, j strictly increases their payoff through this deviation.

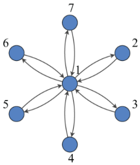
One-Way Flow Model

- Therefore, we can show that every other agent will have an incentive to either link with i or to observe them through a sequence of links (i.e., the network is connected).
- Moreover, the network must be minimally connected.
- If it is not, then there are two paths between a pair of individuals and a player can delete a link and still observe all the players, which would contradict the optimality of actions in a Nash equilibrium.

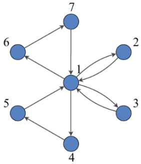
One-Way Flow Model



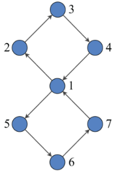
(a) Cycle



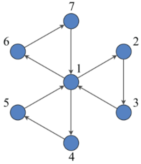
(b) Star



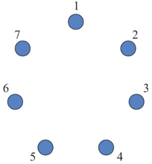
(c) Hub with petals and spokes



(d) Hub with petals



(e) Windmill



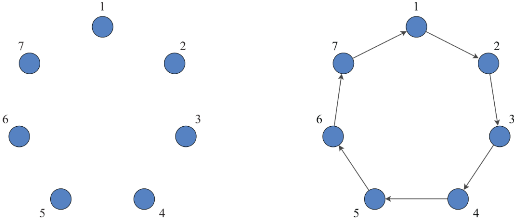
(f) Empty

One-Way Flow Model

- The number of Nash networks increases quite rapidly with the number of players.
- Thus the Nash equilibrium is a fairly permissive requirement.
- Is there some way to restrict the set of networks further based on individual incentives alone?
- Let us consider strict Nash equilibrium.
- It turns out that the requirement of strictness is powerful and eliminates all but two network architectures as candidate networks in our game.

One-Way Flow Model

- In a NE, if two players i and j have a link with the same player l, then player i will be indifferent between forming a link with l and forming a link with j.
- This means that every player has one and only one player who initiates a link with them; thus a nonempty strict Nash network has exactly n links.
- We know that a (nonempty) equilibrium network is connected. It can be shown that the cycle is the unique connected (directed) network with exactly n links.



(a) Empty (b) Cycle

One-Way Flow Model

- In summary, in the one-sided model with one-way flow, a Nash equilibrium network is either connected or empty.
- A strict Nash network is either a cycle containing all players or the empty network.
- For linear payoff function, if $k < 1$ (when the costs are smaller than stand-alone benefits), the cycle containing all players is a unique strict Nash equilibrium.
- The empty network is a unique strict Nash equilibrium when $k > n - 1$ (when the costs of accessing everyone are larger than the benefits,).
- For k in between, the cycle and the empty network are strict Nash equilibria.

One-Way Flow Model

- How about Aggregate efficiency?
- For the class of connected networks, every such network contains at least n links.
- The cycle network containing all players has n links, so it must maximize aggregate payoffs in the class of connected networks
- As the cycle is the only connected network with n links, we know that if a connected network is efficient, it must be a cycle.
- Moreover, if an efficient network contains some links, it must be connected (i.e., a partially connected network with multiple components is never efficient).
- Thus an efficient network is either empty or the cycle.

One-Way Flow Model

- In summary, if $k > n - 1$, the unique efficient network is empty network.
- If $0 < k < n - 1$ (cycle is possible), unique efficient network is cycle.
- Recall when $1 < k < n - 1$, an efficient network is a cycle, while the empty network is also a (strict) Nash equilibrium.
- Thus there is the possibility of coordination failure: individuals may create an empty network even though they could create a connected network in equilibrium.

Two-Sided Links

- Now, consider the network formation in which a link between two players requires the approval of both of them.
- In such games, for any pair of individuals, it is always a best response for each of them to offer to form no link if the other does so.
- Consider a link announcement game. Every player announces a set of intended links.
- Now, $s_{i,j} = 1$ means player i intends to form a link with player j.
- Define $g_{ij} = \min\{s_{i,j}, s_{j,i}\}$.
- The strategy profile s therefore induces a corresponding undirected network $g(s)$.

Two-Sided Links

- Let $n_i(g)$ be the benefit that player i receives from each player that they access through an undirected path in the network.
- Let $\eta_i(g)$ be the number of links they form.
- We still consider linear payoffs $\prod_i(g) = n_i(g) - \eta_i^d(g)k$.
- It is easy to see that empty network is a Nash equilibrium: if every player announces that they want to form no links, then a best response of player i is to announce that they want to form no links as well.

Two-Sided Links

- A network g is pairwise stable if
  1. For every $g_{ij} = 1$, $\prod_i(g) \geq \prod_i(g - g_{ij})$ and $\prod_j(g) \geq \prod_j(g - g_{ij})$.
  2. For $g_{ij} = 0$, $\prod_i(g + g_{ij}) > \prod_i(g) \Rightarrow prod_j(g + g_{ij}) > \prod_j(g)$
- The first condition requires that every link in a stable network must be weakly profitable for the players involved in the link.
- The second condition requires that for every link that is not present in the network, it must be the case that if one player strictly gains from the link, then the other player must be strictly worse off.
- Suppose that the payoffs are linear. (1) If $k < 1$, then a pairwise stable network is minimally connected and if $k > 1$ then the unique pairwise stable network is empty. (2) If $k < n/2$, then an efficient network is minimally connected, while if $k > n/2$, then the efficient network is empty.

Two-Sided Links

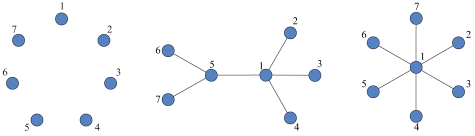
- First, observe that there cannot be two paths between any two players in a pairwise stable network.
- If not, a player could strictly increase their payoff by deleting a link that retained the connectivity of a component.
- Thus a pairwise stable network must be acyclic.
- Moreover, the same logic applies to efficient graph.

Two-Sided Links

- Next, we argue that a pairwise stable network is either empty or connected.
- To see why this is true, consider a nonempty network that is pairwise stable but has multiple components. Let C_1 be the largest component.
- As this is not a singleton and it is acyclic, there is a player i (leaf) who has a single link with a player j: the payoff of i is $|C_1| - k \geq 1$.
- Now consider a player, l, who lies outside component C_1. We can see l has incentive to form link with j, which yield a net marginal benefit of $|C_1| - k > |C_1| - k - 1 \geq 0$.
- Moreover, j would not be worse off with the link. But then the original network g would not be pairwise stable.
- It is clear if $k < 1$, hen every pair of players has an incentive to access each other: a pairwise stable network must be connected. If $k > 1$, then no player would be willing to form a link with an isolated player.

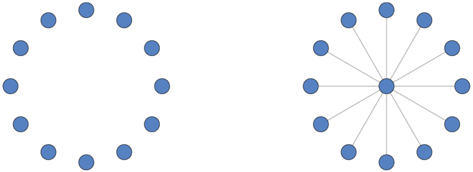
Two-Sided Links

- Let us next describe the efficient networks in this model. Note that a minimally connected network with n nodes has exactly n - 1 links.
- Consider a nonempty network with two minimal components with $l \geq 2$ and $m \geq 2$ nodes. The aggregate payoffs are $l^2 - 2(l-1)k$ and $m^2 - 2(m-1)k$.
- As the network is efficient, the components must generate payoffs greater than the corresponding empty networks: $l^2 - 2(l-1)k \geq l$ and $m^2 - 2(m-1)k \geq m$.
- If we aggregate the two components to create one minimal component, the total payoffs are $(l+m)^2 - 2(l+m-1)k > l^2 - 2(l-1)k + m^2 - 2(m-1)k$ Given $l^2 - 2(l-1)k \geq l$ and $m^2 - 2(m-1)k \geq m$.
- Thus an efficient network is either minimally connected or it is empty.
- A minimally connected network yields a total payoff of $n^2 - 2(n-1)k$, while the empty network yields total payoffs equal to n. Thus the connected network dominates the empty network if and only if $k < n/2$.



(a) Empty: $k > 1$ (b) Two-hubs: $k < 1$ (c) Star: $k < 1$

Figure 3.4
Pairwise stable networks: two-sided link model.



(a) Empty: $k > n/2$ (b) Star: $k < n/2$

Figure 3.5
Efficient networks: two-sided links model.

Figure: [Goyal, 2023]

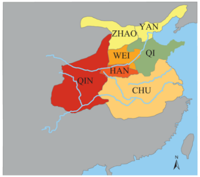
Wars

- War and violent conflict are recurring themes and continue to be important in the twenty-first century
- Larger wars have generally brought multiple opponents into play, and alliances play a central role in such wars.
- We can use network analysis to understand how the patterns of physical contiguity shape wars, how alliances affect the belligerence of different parties, and what the incentives to create alliances are.

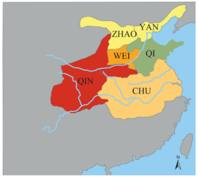
Wars

- Network induces complex interplay of positive and negative strategic effects among players.
- For example, when players A and B form an alliance, they hope to support each other —possibly by sharing resources and information.
- Thus an alliance may strengthen the position of both A and B vis-á-vis other opponents.
- This benefit comes with a potential downside: the effort of A benefits A, but it also benefits B. This spillover makes A's effort a public good and can lower the incentives of A to exert effort.
- An alliance between A and B will have effects on other opponents: they may be obliged to raise their efforts in the face of such an alliance.
- Moreover, in large-scale conflicts, it may be the case that A and B are in an alliance, while B is in an alliance with other players, X and Y.

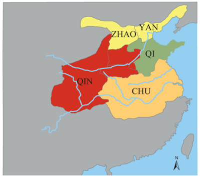
Wars



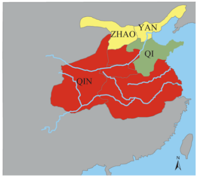
350 BC



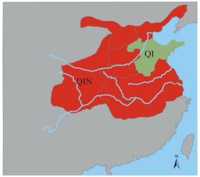
230 BC



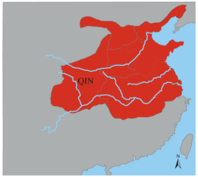
225 BC



223 BC



222 BC



221 BC

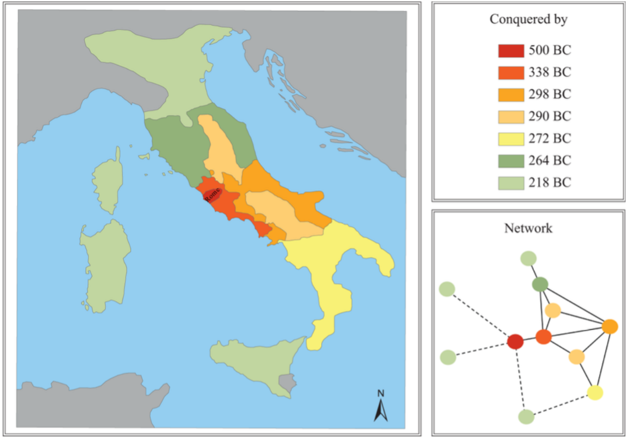


Figure 10.11
Expansion of the Roman republic, 500 BC–218 BC. *Source*: Scarre (1995).

Wars

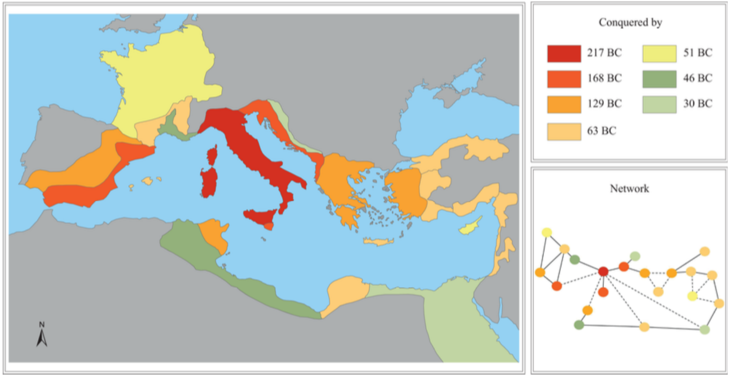


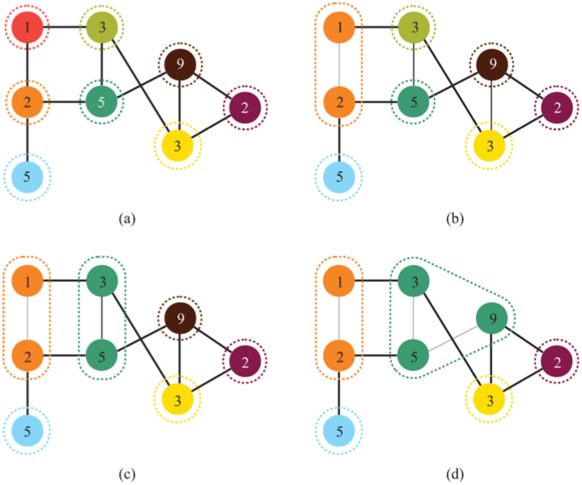
Figure 10.12
Expansion of the Roman republic, 217 BC–30 BC. *Source*: Wittke, Olshausen, Szydlak et al. (2010).

Wars

- We investigate a Dynamic Model of Wars and Conquest in which rulers seek to maximize the resources they control by waging war and capturing new territories.
- There is a set of nodes $V = \{1, 2, .,, , n\}$, with $n \geq 2$. The nodes are connected in a network, represented by an undirected graph g.
- Every node $i \in V$ is controlled/owned by one ruler. Let $\phi : V \to R$ denote the ownership function.
- The resources of rule i under ϕ is $R_i(\phi) = \sum_{v \in \phi^{-1}(i)} r_v$
- A link between two nodes signifies access. (For example, access may reflect physical contiguity).
- When two rulers fight, the probability of winning is modeled through Tullock contest function: $p(x, y) = \frac{x^\gamma}{x^\gamma + y^\gamma}$. γ captures technology.

Wars

- The game takes place in discrete time.
- At the start of a round, one of the rulers is picked with equal probability from the set of remaining rulers.
- The chosen ruler, such as i, chooses either to be peaceful or to attack one of their neighbors.
- If a ruler attacks a rival, they do so with all their current resources.
- If they choose peace, one of the remaining rulers is asked to choose between war and peace, and so forth. If no ruler chooses war, the game ends.
- If the attacker loses, the round ends. Otherwise, the attacker is allowed to attack neighbors until they lose, choose to stop, or there are no neighbors left to attack.
- When two rulers i and j fight, the winner takes over the entire kingdom of the loser. For simplicity, we assume that there are no losses or costs of war.



1
3
9
2
5
2
3
5
(a)
(b)
(c)
(d)

Wars

- The strategy of ruler i is, based on the current state (ownership and set of ruler picked before i), chooses a sequence of rulers to attack. The equilibrium is Markov perfect equilibrium.
- Let us first consider the incentives to wage war.
- A ruler picked to fight needs to decide whether to fight or to remain peaceful, and if fighting is desirable, then to decide whom to attack.
- This depends on the γ in the contest function:
- Rich rewarding: If $\gamma > 1$, then $(x + y)p(x, y) > x$, the expected resources of the richer player are higher than their current resources and the expected resources of the weaker player are lower.
- Poor rewarding: If $\gamma < 1$, then $(x + y)p(x, y) < x$.
- This means that if rulers have unequal resources and are myopic, no peace is possible. In the game, rulers are farsighted and care only about the long-run outcome. In this setting, a ruler may decide not to fight a neighbor, as that would bring them in contact with a more powerful ruler.

Wars

- To develop some intuition, consider a simple case.
- Suppose that three rulers, located in a complete network, have equal resources given by x.
- Let $\gamma = 0$. If two rulers have fought, then the state must contain one ruler with resources 2x and the other ruler with resources x. It follows that the poorer ruler has a strict incentive to wage a war.
- Anticipating this, consider the incentives of rulers at the initial state with three active rulers. As rulers have equal resources and the network is complete, all three rulers have the same incentives.
- As the probability of surviving two wars is $1/4$, the expected payoff from waging a war is $3x/4$. This tells us that there are no wars in equilibrium.

Wars

- By contrast, consider very large γ.
- When there are two rulers, one of them must have 2x resources and the other x. The ruler with more resources wins a war with probability close to 1, and therefore they expect to increase payoffs.
- Anticipating this order of moves in the two-ruler state, at the initial state, all three rulers have a strict incentive to wage war.
- This is because at the initial state, the expected payoff on waging a war is 3x/2, which is larger than the expected payoff from no one fighting.
- We see that with large γ, rulers will wage war, leading to a hegemony.

Wars

- The role of resources is also important.
- As before, for simplicity, consider three rulers linked to each other.
- Suppose that resources are very unequal: for example, rulers 1 and 2 have equal resources, x, and ruler 3 has resources 3x.
- When $\gamma = 0$, the two poorer rulers now wish to fight, while the rich ruler does not. The outcome is war and hegemony.
- When γ is very large. Now ruler 3 will win any war they fight, so they have a strict incentive to fight two wars. The outcome will be the hegemony of ruler 3.

Wars

- Based on the simple case, let me show the formal equilibrium.
- Define active rulers at ϕ as: $Act(\phi) = \{i \in R : \emptyset \subsetneq \phi^{-1} \subsetneq V\}$; they are who controls at least one vertex but does not control all vertices.
- Under rich rewarding contest success function, Suppose g is a connected network. In the equilibrium, ruler chooses to attach a neighbor if $|Act(\phi)| \geq 3$, and at least one of the active rulers attacks their opponent if $|Act(\phi)| = 2$. The outcome is hegemony, and the probability of becoming a hegemon is unique for every ruler.
- The result predicts incessant fighting, preemptive attacks, and long attacking sequences for all rich rewarding contest functions, any connected network, and generic resources.

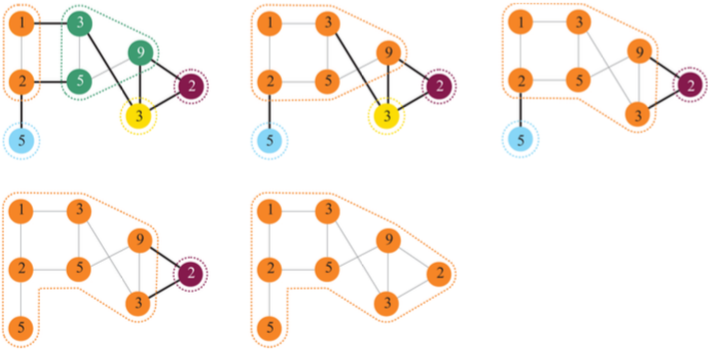


Figure 10.15
Full attacking sequence (f.a.s.).

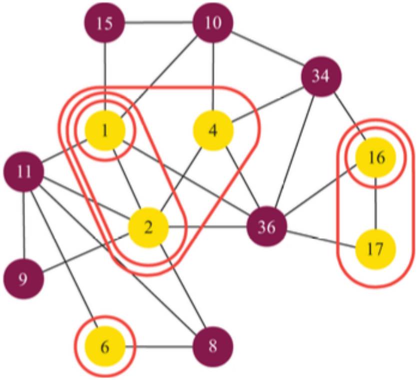


Figure 10.16
Weak rulers (surrounded by thick red lines) and strong rulers.

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