Lecture: Equilibrium Refinements in Extensive Form Games

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Note: This lecture note is adapted from *Rationality in extensive Form Games* by Andres Perea.

1 Introduction

In this note, we only consider finite extensive-form game with perfect recall. This includes perfect and imperfect information extensive-form game. This also includes incomplete information game because incomplete information can be modeled as a kind of imperfect information with a change move by Harsanyi.

2 Decision-theoretic Framework

2.1 Subjective Expected Utility

Consider a decision maker who faces the problem of reaching a decision under uncertainty. Assume that the consequence of a decision depends upon uncertain events that are beyond the decision maker's control. We distinguish two types of uncertain events here. The first type contains random events for which the probabilities of the possible outcomes are *objectively unknowns.* One may think of throwing a dice, or spinning a roulette wheel. Such events are called *subjective unknowns.* The second type contains uncertain events to which no obvious objective probabilities can be assigned. Think, for instance, of events that are the result of a decision making process by other people. These events are called subjective unknowns.

Noe, let us formally define objective and subjective unknowns.

Let C be some finite set, representing the consequences that can possibly be induced by the decisions available. Let $\triangle C$ be the set of objective probability distributions on C, that is,

$$\triangle C = \{p : C \to [0,1] | \sum_{c \in C} p(c) = 1\}$$

Let Ω be some finite set of states, representing the subjective unknowns relevant for the decision problem. The fact that the consequence of a decision depends on the realization of objective and subjective unknowns is modeled by identifying each decision with some function $f : \Omega \to \triangle C$. Particularly, $f(\omega)(c)$ represents the objective probability that c will happen if decision the realized state is ω.

F is the set of acts.

Definition 1. We say that the preference relation $\succeq$ on F is of the subjective expected utility type if there exists some probability distribution $\mu \in \triangle(\Omega)$ and some function $u : C \to \mathbb{R}$ such that for every $f, g \in F$ it holds : $f \succeq g$ if and only if

$$\sum_{\omega\in\Omega} \mu(\omega) \sum_{c\in C} f(\omega)(c)u(c) \geq \sum_{\omega\in\Omega} \mu(\omega) \sum_{c\in C} g(\omega)(c)u(c)$$

Here, μ is called a subjective probability distribution on the set of states, whereas u is called a utility function on the set of consequences. We call

$$\mathbb{E}_{\mu,u}(f) = \sum_{\omega\in\Omega} \mu(\omega) \sum_{c\in C} f(\omega)(c)u(c)$$

is the subjective expected utility of act f generated by μ and u.

Theorem 1. *Let the preference relation $\succeq$ be of the subjective expected utility type, and let there be some $f, g \in F$ with $f \succ g$. Let $\succeq$ be represented by some subjective probability distribution $\mu \in \triangle(\omega)$ and some utility function $u : C \to \mathbb{R}$. Let $\overline{\mu}$ and $\tilde{u}$ be such that they also represent $\succeq$. Then, $\tilde{\mu} = \mu$ and there exist numbers $A > 0$, $B \in \mathbb{R}$ such that $\tilde{u}(c) = Au(c) + B$ for all $c \in C$.*

2.2 Conjectures about Opponents' Behavior

We let H_i be the information set for player i and $A(h)$ be the set of action at $h \in H$.

Definition 2 (Strategy)**.** A strategy for player i is a function $s_i : H_i \to A$.

In an extensive form game, the possible consequences of a strategic interaction among the players are represented by the set Z of terminal nodes in the game tree. Therefore, $C = Z$; For each player i, there is a utility function $u_i(z)$ attached for each terminal z.

In the game, players have some conjecture about other players' strategy. Therefore, $\omega = \times_{j\neq i} S_j$. We use $\mu^i(s_{-i})$ to denote some "measure of likelihood" that player i attaches to the strategy profile s_{-i}. The subjective probability distribution $\mu^i \in \triangle S_{-i}$ is called player $i's$ conjecture about the opponents' strategies.

The objective uncertainty is determined by the chance move. Given "state" $\omega = s_{-i}$ and objective chance move, they induce the probability of each outcome $c = z$, denoted by $\mathbb{P}(s_i, s_{-i})$.

Then, the subjective expected utility for a player is:

$$\mathbb{E}(s_i) = \sum_{s_{-i}\in S_{-i}} \mu^i(s_{-i}) \sum_{z\in Z} \mathbb{P}_{(s_i,s_{-i})}(z)u_i(z)$$

Player $i's$ preferences over his strategies can thus be characterized by: $s_i \succeq_i s_i'$ if and only if $\mathbb{E}(s_i) \geq \mathbb{E}(s_i')$.

Except for the concept of rationality, we impose following assumptions:

The first assumption is that the probability distribution $\mu^i \in \triangle S_{-i}$ can be written as the product of its marginal distributions on S_j. We refer to $\mu^i_j(s_j)$ as player $i's$ conjecture about player $j's$ strategy choice.

Assumption 1. *For every player i, the conjecture μ^i is such that for every $s_{-i} = (s_j)_{j\neq i} \in S_{-i}$ it holds: $\mu^i(s_{-i}) = \prod_{j\neq i} \mu^i_j(s_j)$*

The second assumption is that the conjectures held by different players about player k's strategy should coincide.

Assumption 2. *For all triples $i, j, k \in I$ of of pairwise different players, we have $\mu_k^i = \mu_k^j$.*

By the assumptions above, we may define, for each player i, the probability distribution $\mu^i \in \triangle S_i$, representing the players' common conjecture about player $i's$ strategy choice. Hence, $\mu_i = \mu_i^j$ for all players $j \neq i$. Such common conjectures are called mixed conjectures about player i.

Definition 3. A mixed conjecture about player i is a probability distribution $\mu^i \in \triangle S_i$.

Definition 4. A behavioral conjecture about player i is a vector $\sigma_i = (\sigma_{ih})_{h \in H_i}$ where $\sigma_{ih} \in \triangle A(h)$ for every $h \in H$.

For every $h \in H_i$ and every action $a \in A(h)$, the probability σ_{ih} is to be interpreted as the players' common subjective probability assigned to the event that player i would choose action a at h.

Player i holds initial conjecture σ_i. However, during the game, they may need to revise their conjecture. See Figure 1. Assume that player 2 holds the behavioral conjecture $\sigma_1 = c$. If player $2's$ information set is reached, then player 2 knows that his initial conjecture about player $1's$ behavior has been contradicted, and should thus revise this conjecture. We call revised conjectures about the opponents' past play that players undertake at their own information sets are called **beliefs**.

Formally, a belief for player i should specify at each of player $i's$ information sets a conjecture about the opponents' past play that has led to this information set. At a given information set, this conjecture somehow reflects player $i's$ "personal theory" about what happened in the past. Suppose that the game has reached some information set $h \in H_i$ and that player i is asked to form a conjecture about the opponents' past play. Since player i knows that one of the nodes in h has been reached, and each node in h can be identified with the path, or past play, that leads to this node, such a conjecture about past play may be represented as a subjective probability distribution β_{ih} on the set of nodes in h. This leads to the following definition of a belief.

Definition 5 (Belief)**.** A belief for player i is a vector $\beta_i = (\beta_{ih})_{h \in H_i}$ where $\beta_{ih} \in \triangle h$ for every $h \in H_i$. A profile $(\beta_i)_{i \in I}$ of beliefs is called a belief system.

The combination of a behavioral conjecture profile $(\sigma_i)_{i \in I}$ and a belief system $(\beta_i)_{i \in I}$ specifies, for each player i and each information set $h \in H_i$, player $i's$ conjecture about past play up to h (given by β_{ih}) and player $i's$ conjecture about future play after h (given by σ_{-i}). We refer to such combinations as **assessments**.

Definition 6 (Assessments)**.** An assessment is a pair (σ, β) where $\sigma = (\sigma_i)_{i \in I}$ is a behavioral conjecture profile and $\beta = (\beta_i)_{i \in I}$ is a belief system.

Example 1. *Let (σ, β) be the assessment where $\sigma_1 = (c, 0.2f + 0.8g, 0.8h + 0.2i)$, $\sigma_2 = (0.5d + 0.5e)$ and $\beta_2 = (0.3, 0.7)$.*

In general, an assessment (σ, β) allows us to determine, for every player i and every information set $h \in H_i$, the expected utility at h generated by the different player i strategies.

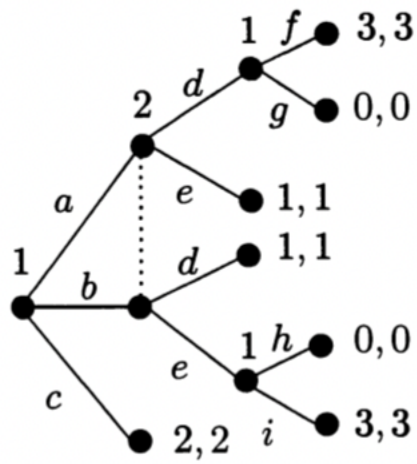

Figure 1: Assessment

2.3 Best response

Given an assessment (σ, β), which behavior by the players may be viewed optimal, given this assessment? We start to introduce three types of best response.

Let $u_i(s_i, \sigma_{-i})$ be the expected utility of S_i, given player $i's$ conjecture σ_{-i} about the opponents' behavior. Note that $u_i(s_i, \sigma_{-i})$ only depends upon the initial mixed conjectures induced by σ , and not upon the way conjectures are revised in σ.

Definition 7. A strategy s_i is called a best response against σ if $u_i(s_i, \sigma_{-i}) = \max_{s'_i \in S_i} u_i(s'_i, \sigma_{-i})$.

The notion of best response selects strategies that are optimal "at the beginning of the game". However, a best response may no longer be optimal once the initial conjectures in σ have been contradicted by the play of the game.

Let $H_i(s_i)$ is the collection of player i information sets that are not avoided by s_i. Also let $u_i[(s_i, \sigma_{-i})|h, \beta_{ih}]$ be the expected utility of choosing s_i at information set h, given the beliefs β_{ih} and given the conjecture σ_{-i} about the opponents' future behavior.

Definition 8. A strategy s_i is called a sequential best response against the assessment (σ, β) if at every information set $h \in H_i(s_i)$ we have that $u_i[(s_i, \sigma_{-i})|h, \beta_{ih}] = \max_{s'_i \in S_i(h)} u_i[(s'_i, \sigma_{-i})|h, \beta_{ih}]$

A sequential best response thus constitutes a best response at each of player $i's$ information sets that are not avoided by it.

Under some "consistency conditions" on the beliefs β, each sequential best response is also a best response in the definition 7. The consistency condition basically states that at every information set $h \in H_i$, the beliefs β_{ih} should be compatible with player $i's$ conjecture σ_{-i} about the opponents' behavior, whenever h is reached with positive probability under σ. In this case, we say that the assessment is Bayesian consistent.

For a given nodes x, let $\mathbb{P}_\sigma(x)$ be the probability that x is reached under σ.

Definition 9. As assessment (σ, β) is called Bayesian consistent if for every player i and every information set $h \in H_i$ with $\mathbb{P}_\sigma(h) > 0$ it holds that $\beta_{ih}(x) = \frac{\mathbb{P}_\sigma(x)}{\mathbb{P}_\sigma(h)}$ for all odes $x \in h$.

Lemma 1. *Let (σ, β) be a Bayesian consistent assessment and s_i a sequential best response against (σ, β). Then s_i is a best response against σ.*

The third best response is local. In contrast to the notions of best response and sequential best response, local best response is a criterion which judges the optimality of actions rather than strategies.

Let σ_{-h} be the restriction of σ on information sets other than h. Let $u_i[(a, \sigma_{-h})|h, \beta_{ih}]$ be the expected utility of choosing action a at h, given the conjecture σ_{-h} about the behavior at information sets other than h (including the conjecture about player $i's$ behavior at such information sets), and the beliefs β_{ih}.

Definition 10. Let (σ, β) be an assessment, $h \in H_i$, and a an action at h. Then, we say that a is a local best response against (σ, β) if

$$u_i[(a, \sigma_{-h})|h, \beta_{ih}] = \max_{a'_i \in A(h)} u_i[(a'_i, \sigma_{-h})|h, \beta_{ih}]$$

3 Solution Concepts

A criterion of rational decision making in extensive form games should give an answer to the following two questions: (1) for a given assessment (σ, β), which strategies may be viewed acceptable?, and (2) which assessments (σ, β) may be viewed reasonable?

The answer for (1) is that, for a given assessment, players are expected to choose either a best response or a sequential best response against this assessment, depending on the concept.

The second question, however, is the most interesting one, and we aim to explore rationality criteria that give different answers to the question which assessments are "reasonable".

3.1 Backward Induction

In this subsection we focus on a special class of games, namely that in which at each point in time the player knows perfectly which actions and chance moves have occurred until that moment. Formally, this means that every information set contains exactly one node. Such games are called games with perfect information.

Since the players' beliefs β_I are trivial in a game with perfect information, the players' conjectures about the opponents' behavior at any point in the game are completely described by a behavioral conjecture profile σ.

Backward induction imposes that at every node x in the game, σ should only assign positive probability to those actions at x which are local best responses against σ.

Definition 11 (Backward Induction)**.** Let Γ be a game with perfect information and σ a behavioral conjecture profile. Then, σ is said to satisfy backward induction if for every player i and every decision node x controlled by player i, $\sigma_{ix}(a) > 0$ only if a is a local best response against σ.

A strategy s_i is called a backward induction strategy if there is a behavioral conjecture profile σ satisfying backward induction such that s_i is a sequential best response against σ.

In the definition, when we say best response against σ, we should think it is best response against (σ, β) where β is the trivial belief system in a game with perfect information.

3.1.1 Property

How to find this strategy? Behavioral conjecture profiles satisfying backward induction can easily be found by means of the iterative procedure, which is called the backward induction procedure. By applying the procedure, you will easily see that:

Lemma 2. *For every game with perfect information there is some pure behavioral conjecture profile σ which satisfies backward induction.*

We can have a more stronger uniqueness result if Γ is in generic position. Generic position refers to that in a game with perfect information, there are no chance moves, and that for every player i and every pair of different terminal nodes $u_i(z) \neq u_i(z')$. Then, with usual backward induction procedure, we get

Lemma 3. *Let Γ be a game with be a game with perfect information in generic position. Then, there is a unique behavioral conjecture profile σ which satisfies backward induction. Moreover, this σ is pure.*

Recall in the definition of backward induction, we focus on action a is a local best response. The following theorem states an important property of backward induction: sequential best response. If σ satisfies backward induction, then every behavioral conjecture σ_i assigns only positive probability to those player i strategies that are best responses against σ: $\sigma_i(s_i) > 0$.

Theorem 2. *Let Γ be a game with perfect information and σ be a behavioral conjecture profile which satisfies backward induction. Then, for every player i, we have that $\sigma_i(s_i) > 0$ only if s_i is a sequential best response against σ and also the best response against σ.*

The best response comes from the trivially Bayesian consistent.

We should note that player i assigns positive probability only to those strategies that are best responses for player i also for the **revised** conjectures about player i. Let Γ_x be a subgame starting at x. The notation with x should be think as starting from node x, in the following theorem.

Theorem 3. *Let Γ be a game with perfect information and σ a behavioral conjecture profile which satisfies backward induction. Then, for every node x and every strategy $s_i \in S_i^x$ we have that $\sigma_i^x(s_i) > 0$ only if s_i is a best response against σ^x in the subgame $_x$.*

This theorem follows immediately from the previous one.

3.1.2 Comment

In view of the above theorem, the concept of backward induction assumes that at every subgame, all players believe that their opponents will choose best responses against σ in the sequel. In particular, if a subgame can only be reached through a non-best response by player i, then $i's$ opponents should still believe that player i will choose a best response against σ in this subgame. See example in the figure 2.

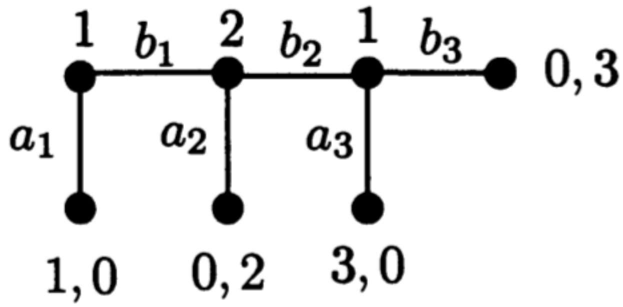

Figure 2: Problem of backward induction

3.2 Nash Equilibrium

Recall the subjective utility function:

$$\mathbb{E}(s_i) = \sum_{s_{-i} \in S_{-i}} \mu^i(s_{-i}) \sum_{z \in Z} \mathbb{P}_{(s_i, s_{-i})}(z) u_i(z)$$

We assume players' utility function $u_i : Z \to \mathbb{R}$ are common knowledge. Then, the only uncertainty is the subjective probability distributions $(\mu_i)_{i \in I}$ held by the players.

Now, if we assume these subjective probability distributions would also be common knowledge, then player i would be informed about the conjecture μ_i held by the opponents about player i, and each of player $i's$ opponents would be informed about the conjecture μ_{-i} held by player i about the other players' behavior.

Consequently, if the opponents believe that player i chooses a best response, then the opponents' common conjecture μ_i about player i should assign positive probability only to those player i strategies s_i that are best responses against μ_{-i}. Mixed conjecture profiles with this property are called Nash equilibria.

Definition 12. A mixed conjecture profile $\mu = (\mu_i)_{i \in I}$ is called a Nash equilibrium if for every player i, $\mu_i(s_i) > 0$ only if s_i is a best response against μ.

A strategy s_i is called a Nash equilibrium strategy if there is a Nash equilibrium μ such that s_i is a best response against μ.

3.2.1 Property

We can have a similar definition for the behavioral conjecture profile. For a given behavioral conjecture profile σ, we refer to $u_i(\sigma) = \sum_{s_i S_i} \sigma_i(s_i) u_i(s_i, \sigma_{-i})$ as the expected utility for player i induced by σ. Nash equilibria may now be characterized as follows.

Lemma 4. *A behavioral conjecture profile $\sigma = (\sigma_i)_{i \in I}$ is a Nash equilibrium if and only if for every player i, $u_i(\sigma) = \arg\max_{\sigma_i'} u_i(\sigma_i', \sigma_{-i})$.*

Note: in contrast to backward induction, the players' revision of conjectures does not play a role in the concept of Nash equilibrium.

Now we consider the existence by Kakutani's Fixed Point Theorem.

Theorem 4. *For every (finite) extensive form game, there exists a Nash equilibrium $\mu = (\mu_i)_{i \in I}$.*

We do have many more existence results that relax assumptions.

Now, let focus on game Γ be a game with perfect information. Let σ be a behavioral conjecture profile satisfying backward induction. Then, by Theorem 2 and 3, we know σ is then a Nash Equilibrium, even for each subgame. Moreover, Lemma 2 guarantees that we can always find a pure behavioral conjecture profile satisfying backward induction, it follows that a game with perfect information always contains a Nash equilibrium in pure behavioral conjectures.

Lemma 5. *Every game with perfect information contains a Nash equilibrium σ in pure behavioral conjectures. Moreover, this Nash equilibrium σ can be chosen such that in every subgame Γ_x the induced behavioral conjecture profile σ^x constitutes a Nash equilibrium in Γ_x.*

3.2.2 Comment

It should be clear for you: In a game with perfect information, not every Nash equilibrium satisfies backward induction, however. The fundamental difference between both concepts is that Nash equilibrium does not require players to believe that their opponents choose optimally at every subgame, whereas backward induction does.

3.3 Subgame Perfect Equilibrium

Subgame Perfect Equilibrium may be viewed as a combination of backward induction and Nash equilibrium. By Theorem 3 we know that in games with perfect information, a behavioral conjecture profile satisfying backward induction induces, in every subgame, a Nash equilibrium of that subgame. The idea of subgame perfect equilibrium is to extend this property to games without perfect information.

Definition 13. A behavioral conjecture profile σ is called a subgame perfect equilibrium if for every subgame Γ_x, the induced behavioral conjecture profile σ^x is a Nash equilibrium in Γ_x.

3.3.1 Property

By construction, every subgame perfect equilibrium is a Nash equilibrium since the whole game Γ is a subgame.

Lemma 6. *In a game with perfect information, a behavioral conjecture profile σ is a subgame perfect equilibrium if and only if it satisfies backward induction.*

Theorem 5. *Every game in extensive form has at least one subgame perfect equilibrium.*

3.3.2 Comment

As with backward induction, in a subgame perfect equilibrium σ players are required to believe, in every subgame, that their opponents will choose best responses against σ in this subgame. In particular, they should believe so even if this subgame could only have been reached via non-best responses by the same opponents against σ. As with backward induction, the question may be raised why σ should still constitute a Nash equilibrium in this subgame if the event of reaching this subgame has contradicted player 2's conjecture that player 1 will choose optimally against σ.

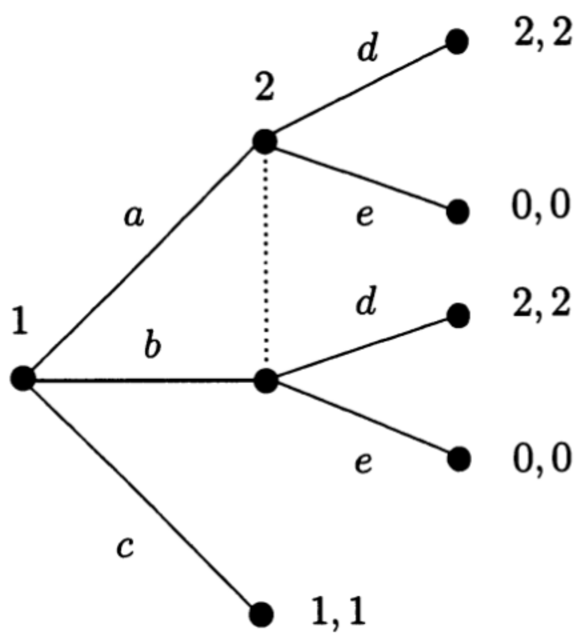

Figure 3: Problem of SPE

One could say that the concept of subgame perfect equilibrium requires the players to "ignore" the his- tory that has led to a particular subgame, and that each subgame should be treated as if the game would actually start here.

See next subsection for another problem.

3.4 Perfect Equilibrium

In the concepts of Nash equilibrium and subgame perfect equilibrium, players are required to attach positive probability only to those opponents' strategies that constitute best responses. However, players may still assign positive probability to opponents' strategies that are not local best responses. Consider the following game in the figure 3 where $\sigma = (c, e)$ is a subgame perfect equilibrium (and hence a Nash equilibrium), but e is not a local best response for any beliefs β for player 2.

The reason why e is a best response against σ but not a local best response lies in the fact that player $2's$ conjecture $\sigma_1 = c$ excludes the event that player $2's$ information set will be reached. Consequently, player 2 is indifferent between his two actions. How will you solve this problem?

How about this: If player 2 would attach a small positive probability to the actions a and b, then e would no longer be a best response, and the "unreasonable" conjecture $\sigma_2 = e$ could thus be eliminated. This is the basic idea underlying the concept of perfect equilibrium (Selten, 1975).

Now, we will see the power of β. Say that a behavioral conjecture σ_i about player i is strictly positive if $\sigma_{ih}(a) > 0$ for every $h \in H_i$ and every action $a \in A(h)$. A strictly positive behavioral conjecture thus takes into account all possible actions that may be chosen by player i.

A behavioral conjecture profile σ is called strictly positive if each σ_i is strictly positive. In the literature, it is also called completely mixed. For a given strictly positive behavioral conjecture

		Player II	
		L	*R*
Player I	*T*	1, 1	0, 0
	B	0, 0	0, 0

Figure 4: Example:PE

profile a every information set is reached with positive probability, that is, $\mathbb{P}_\sigma(h) > 0$ for every h. This implies there is a unique belief system β such that (σ, β) is a Bayesian consistent assessment. We refer to β as the belief system induced by σ. We say that the Bayesian consistent assessment (σ, β) is strictly positive if σ is strictly positive.

Definition 14. A behavioral conjecture profile σ is called perfect equilibrium if there is a sequence $(\sigma^n, \beta^n)_{n\in N}$ of strictly positive Bayesian consistent assesments such that
(1)$(\sigma^n)_{n\in N}$ converges to σ and
(2) $\sigma_{ih}(a) > 0$ only if a is a local best response against (σ^n, β^n) for every n.

Hence, in the concept of perfect equilibrium, for the players to attach positive probability to an action a it is not sufficient that a is optimal against σ; the action a should, in addition, remain optimal if σ is replaced by a slightly perturbed conjecture σ that takes all possible actions into account.

Example 2. *See Figure 4. Two pure strategy equilibria:* (T, L) *and* (B, R).
And only (T, L) *is PE.*

3.4.1 Property

No weakly dominated strategy can be a part of a perfect equilibrium.

Theorem 6. *In every perfect equilibrium, every (weakly) dominated strategy is chosen with probability zero.*

Theorem 7. *In every extensive form game there is at least one perfect equilibrium.*

Note that the concept of perfect equilibrium requires players to assign zero probability to suboptimal actions by the opponents against σ^n, but does not explicitly require them to assign zero probability to suboptimal strategies by the opponents against σ. The following theorem, however, will show that the latter property is implied by the former. Moreover, this holds for every subgame, which implies that every perfect equilibrium is a subgame perfect equilibrium.

Theorem 8. *Every perfect equilibrium* σ *is a subgame perfect equilibrium.*

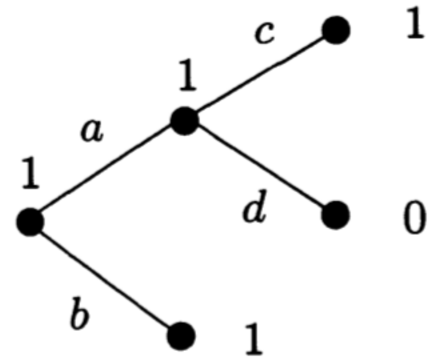

Figure 5: Problem of PE

There is another definition is equivalent to Definition 14 due to Myerson 1978. This definition do not refer to a conventional optimization. We will see it similarity to the proper equilibrium later.

Definition 15. Strategy profile σ^{ϵ} is an ϵ-perfect equilibrium if it is completely mixed, and, for all i and any s_i, if there exists s_i' with $u_i(s_i, \sigma_{-i}^{\epsilon}) < u_i(s_i', \sigma_{-i}^{\epsilon})$, then $\sigma_i^{\epsilon}(s_i) < \epsilon$. A perfect equilibrium σ is any limit of ϵ-perfect strategy profile σ^{ϵ} for some seqence ϵ of positive numbers that converges to 0.

3.4.2 *Normal form perfect equilibrium

Definition 16. A behavioral conjecture profile σ is called a normal form perfect equilibrium if there is a sequence $(\sigma^n)_{n \in N}$ of strictly positive behavioral conjecture profiles such that
(1) $(\sigma^n)_{n \in N}$ converges to σ and
(2) $\sigma_i(s_i) > 0$ only if s_i is a sequential best response against σ^n for every n.

3.4.3 Comment

By construction, the concept of perfect equilibrium requires players to assign probability zero to those actions that are not local best responses against any strictly positive behavioral conjecture profile. More precisely, if player i has to choose at information set h, then he is believed to choose an action a which is not only a local best response against the original conjecture profile σ, but also against some σ close to σ that assigns positive probability to all actions. Here, σ may be interpreted as some slightly perturbed conjecture.

A disturbing feature, however, is the fact that in the perturbed conjecture if, player $i's$ conjecture σ_{-h} about the behavior at other information sets should also assign positive probability to all future actions controlled by player i himself. This restriction seems difficult to justify if we assume that players are confident to carry out correctly their own strategies. See Figure 5. $\sigma(a, c)$ is not a perfect equilibrium, why?

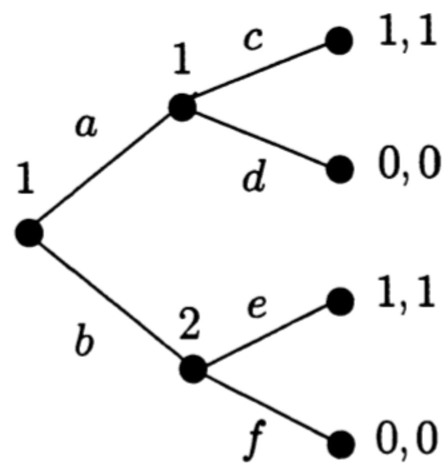

Figure 6: PE and Quasi-PE

3.5 Quasi-Perfect Equilibrium

A possible way to overcome the above drawback is by stating that a player, at each of his information sets, is believed to choose a continuation strategy that is optimal against a slight perturbation of his conjecture about strategy choices not controlled by himself. This is the idea behind the concept of quasi-perfect equilibrium, introduced in van Damme (1984).

Definition 17. A behavioral conjecture profile σ is called a quasi-perfect equilibrium if there is a sequence $(\sigma^n)_{n\in N}$ of strictly positive behavioral conjecture profiles such that
(1) $(\sigma^n)_{n\in N}$ converges to σ and
(2) for every player i, every information set $h \in H_i$ and every $s_i \in S_i(h)$ it holds that $\sigma_{i|h}(s_i) > 0$ only if s_i is a sequential best response against σ^n_{-i} at h for every n.

Each player i is thus believed, at each of his information sets, to solely apply strategies that not only are optimal against the original conjecture profile σ_{-i}, but also against some slight perturbation σ^n_{-i} of σ_{-i} in which player i takes into account all possible actions by the opponents. In contrast to the concept of perfect equilibrium, the perturbation σ^n_{-i} no longer includes player $i's$ own actions.

3.5.1 Property

In fact, there is no general logical relationship between perfect equilibrium and quasi-perfect equilibrium, that is, a perfect equilibrium need not be quasi-perfect, and a quasi-perfect equilibrium need not be perfect.

Example 3. *See Figure 6. Show $((b, c), e)$ is a perfect equilibrium but not a quasi-perfect equilibrium. ($((a, c), e)$ is the unique quasi-perfect equilibrium.)*
$\sigma^n = ((\frac{1}{n}a + (1 - \frac{1}{n}))b, (1 - \frac{1}{n})c + \frac{1}{n}d, (1 - \frac{1}{2n})e + \frac{1}{2n}f)$.

Theorem 9. *Every quasi-perfect equilibrium is a subgame perfect equilibrium.*

Later, we will see every quasi-perfect equilibrium induces a sequential equilibrium, and that every sequential equilibrium constitutes a subgame perfect equilibrium.

Lemma 7. *Every quasi-perfect equilibrium is a normal form perfect equilibrium.*

3.6 Proper Equilibrium

We introduce the concept of proper equilibrium, which is due to Myerson (1978). We then show that every proper equilibrium induces a quasi-perfect equilibrium, implying the existence of quasi-perfect equilibria in every game. This relationship between proper equilibrium and quasi-perfect equilibrium is due to van Damme (1984). Kohlberg and Mertens (1986) have independently proved a weaker version of this theorem, stating that every proper equilibrium induces a sequential equilibrium; a concept which is weaker than quasi-perfect equilibrium.

We say that σ is ϵ-proper if σ is strictly positive and if for every player i and every two strategies $s_i, t_i \in S_i$ with $u_i(s_i, \sigma_{-i}) < u_i(t_i, \sigma_{-i})$ it holds that $\sigma_i(s_i) \leq \epsilon\sigma_i(t_i)$. Hence, if strategy $s_i i$ performs worse than some other strategy t_i, then player $i's$ opponents should deem s_i much less likely than t_i.

Definition 18. Strategy profile σ^ϵ is an ϵ-proper equilibrium if it is completely mixed, and, for all i and any s_i, if there exists s_i' with $u_i(s_i, \sigma_{-i}^\epsilon) < u_i(s_i', \sigma_{-i}^\epsilon)$, then $\sigma_i^\epsilon(s_i) \leq \epsilon\sigma_i^\epsilon(s_i')$. A proper equilibrium σ is any limit of ϵ-proper equilibrium σ^ϵ as ϵ tends to 0.

It is easily shown that every proper equilibrium is a Nash equilibrium.

Example 4. *P357 tf*

3.6.1 Property

Theorem 10. *Every extensive form game has a proper equilibrium.*

Theorem 11. *Let Γ be an extensive form game and σ a proper equilibrium. Then, σ is a quasi-perfect equilibrium.*

4 Consistency and Sequential Rationality

In previous section, we have discussed some restrictions on assessments which, roughly speaking, reflect the requirement that players should believe that opponents act optimally. In the concepts of Nash equilibrium and normal form perfect equilibrium, this requirement is only imposed at the beginning of the game, whereas in other concepts such as backward induction, subgame perfect equilibrium and quasi-perfect equilibrium, it is imposed at every stage of the game. The concepts discussed so far did not explicitly involve the belief system, however, since explicit restrictions were put on the behavioral conjecture profile only. In this subsection we discuss some rationality criteria which impose explicit conditions on the belief system as well.

Recall, given an assessment (σ, β), the behavioral conjecture σ_i specifies at each of player $i's$ information sets $h \in H_i$ the common conjecture held by $i's$ opponents about player $i's$ behavior at h. On the other hand, the belief β_j specifies at each of player $j's$ information sets $h \in H_j$ player $j's$ personal conjecture about the opponents' past play that has led to h. In order for an assessment (σ, β) to be "acceptable", it seems to be a necessary requirement that the behavioral conjectures and beliefs be "consistent", that is, do not contradict one another.

We have introduced Bayesian consistency, which states that the beliefs should be induced by the behavioral conjectures through Bayesian updating, whenever possible. More precisely, if some information set $h \in H_i$ is reached with positive probability under σ, then player i is able to compute, for every node $x \in h$, its probability under σ, conditional on the event that h is reached. Bayesian consistency requires the beliefs β_{ih} at h to coincide with these conditional probabilities.

4.1 Perfect Bayesian Equilibrium

Weak Perfect Bayesian Equilibrium require Bayesian consistency.

For a given assessment (σ, β) and information $h \in H_i$, say that a strategy $s_i \in S_i(h)$ is a sequential best response against (σ, β) at h if

$$u_i[(s_i, \sigma_{-i})|h, \beta_{ih}] = \max_{s_i' \in S_i(h)} u_i[(s_i', \sigma_{-i})|h, \beta_{ih}].$$

Definition 19. An assessment is called sequentially rational if for every player i, every $h \in H_i$ and every $s_i \in S_i(h)$, we have that $\sigma_{i|h}(s_i) > 0$ only if s_i is a sequential best response against (σ, β) at h.

Sequential rationality thus requires the players to believe, at every information set h, that the player at h is playing a sequential best response against (σ, β) at h.

Definition 20. A strategy profile σ of a finite extensive form game is a weak perfect Bayesian equilibrium (weak PBE) if there exists a system of beliefs β such that

(1) σ is sequentially rational given β, and

(2) Bayesian consistent: for all h on the path of play, i.e. $\mathbb{P}_\sigma(h) > 0$ it holds that $\beta_{ih}(x) = \frac{\mathbb{P}_\sigma(x)}{\mathbb{P}_\sigma(h)}$ for all nodes $x \in h$.

4.1.1 Comment and almost perfect Bayesian equilibrium

While weak perfect Bayesian equilibrium does impose sequential rationality everywhere, it is insufficiently demanding in the following sense: It places no restrictions on the system of beliefs on information sets off the path of play. In particular, this means that for games that do not have perfect information, there will be strategy profiles that are weak perfect Bayesian, and yet are not subgame perfect.

For example in the Figure 7, profile $\sigma = (LB, l)$, with beliefs β that assign probability 1 to the node reached by T, is weak PBE but not subgame perfect.

Moreover, Bayesian consistency notion is weak. See Figure 8. Let $\sigma = (a, (d, f))$ and player $2's$ belief $(1, 0)$ and $(0, 1)$ respectively. This assessment is Bayesian consistent since both of player 2's information sets are reached with probability zero under σ. Player $2's$ beliefs do not seem plausible, however, since player $2's$ beliefs at his second information set contradict his beliefs at his first information set. We say that the assessment is not **updating consistent**.

How to solve this problem? Let $\mathbb{P}_{(s_i, \sigma_{-i})}(x|h^1, \beta_{h^1})$ be the probability that the node x is reached, conditional on h^1 being reached and given the beliefs β_{ih^1} at h^1.

Definition 21. An assessment (σ, β) is called updating consistent if for every player i, every two information sets $h^1, h^2 \in H_i$, where h^2 comes after h^1, and every strategy $s_i \in S_i(h^2)$ with $\mathbb{P}_{(s_i, \sigma_{-i})}(h^2|h^1, \beta_{h^1}) > 0$, it holds

$$\beta_{ih^2}(x) = \frac{\mathbb{P}_{(s_i, \sigma_{-i})}(x|h^1, \beta_{h^1})}{\mathbb{P}_{(s_i, \sigma_{-i})}(h^2|h^1, \beta_{h^1})}$$

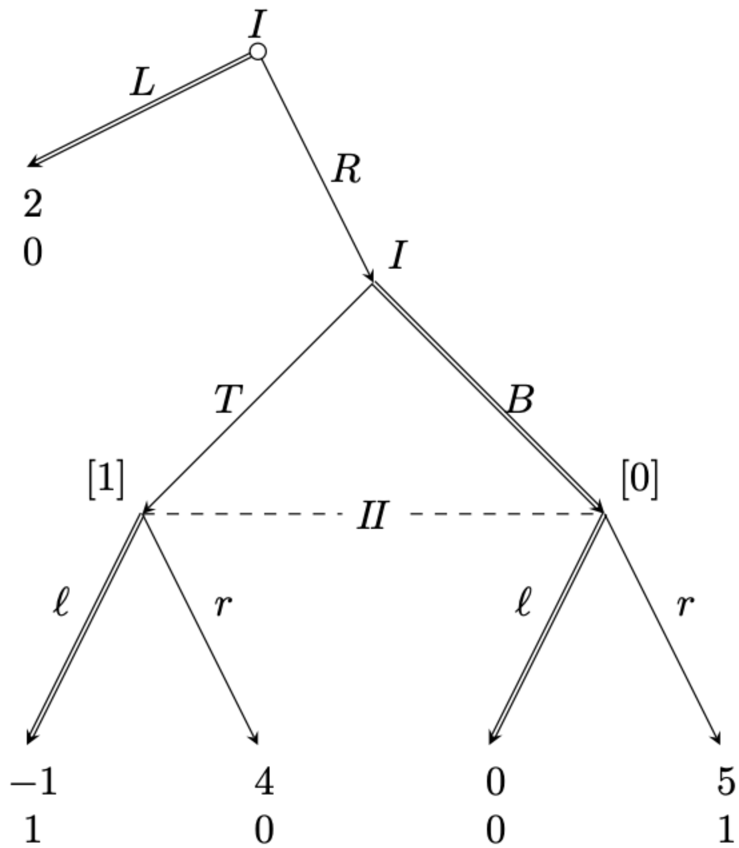

Figure 7: weak PBE

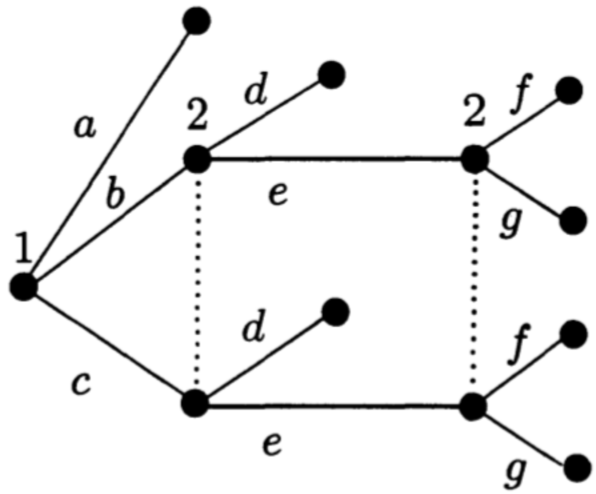

Figure 8: Problem of Bayesian consistency

In the updating consistent, it only consider the following information set of the same player. We can extend it to any following information set. This can solve the problem of weak PBE. Then, at information sets off the equilibrium path, beliefs are determined by Bayes' rule and the players' equilibrium strategies where possible.

Definition 22. A strategy profile σ of a finite extensive form game is an almost perfect Bayesian equilibrium (almost PBE) if there exists a system of beliefs β such that

(1) σ is sequentially rational given β, and

(2) for any information set h^1 and the following information set h^2 reached with positive probability from h^1 under (σ, β), $\beta(x)$ is determined by Bayes' rule according to β_{h^1} and σ.

Theorem 12. *Every almost perfect Bayesian equilibrium is subgame perfect.*

In some cases, almost perfect Bayesian equilibrium is also not strong enough. See Figure 9 where player 1 and 2 choose A and A'. Player 3's belief is not constrained. Do you have any idea to solve this? Actually, we have seen the simialr solution before.

4.2 Sequential Equilibrium

Kreps and Wilson (1982a) proposed a condition on assessments, called **consistency**, that is stronger than both updating consistency and Bayesian consistency. The idea is the following. Suppose that instead of σ the players would hold a behavioral conjecture profile σ in which all possible actions are taken into account, that is, every action is assigned a positive probability. In this case, σ would induce a unique belief system $\tilde{\beta}$ since at every information set a unique belief vector is obtained by Bayesian updating. Moreover, the behavioral conjectures and the beliefs in $(\tilde{\sigma}, \tilde{\beta})$ would be "perfectly compatible" with one another.

Definition 23. An assessment (σ, β) is called consistent if there is a sequence (σ^n, β^n) of strictly positive Bayesian consistent assessments converging to (σ, β).

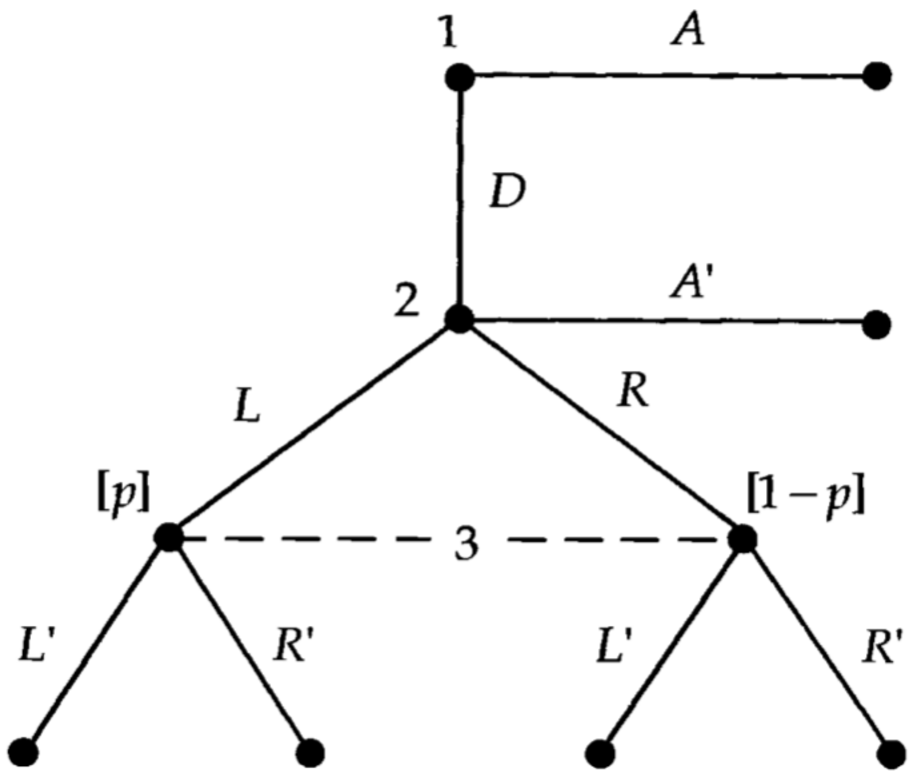

Figure 9: Caption

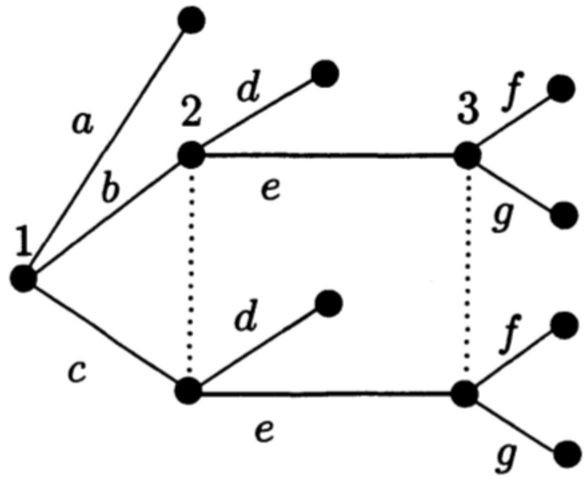

Figure 10: Consistency

The reader may easily verify that every consistent assessment is updating consistent and Bayesian consistent. In order to see that consistency can be a strictly stronger requirement than these two criteria, consider Figure 10. Let $\sigma = (a, d, f)$, player 2 has beliefs $(1, 0)$ and player 3 has beliefs $(0, 1)$. This assessment is Bayesian consistent and updating consistent, but is not consistent.

Some of the concepts, like perfect equilibrium and quasi-perfect equilibrium, implicitly involve the belief system β by considering sequences of strictly positive Bayesian consistent assessments converging to the original assessment, and imposing optimality conditions against assessments in this sequence. In this subsection, we shall present a "strategic" condition on assessments, called sequential rationality, that explicitly uses the belief system.

By simultaneously imposing consistency and sequential rationality, we obtain the concept of sequential equilibrium (Kreps and Wilson, 1982a).

Definition 24. An assessment (σ, β) is called a sequential equilibrium if (σ, β) is consistent and sequentially rational.

4.2.1 Property

Note that the concept of perfect equilibrium and quasi-perfect equilibrium also involves a form of sequential rationality. The strategy σ are bu construction limits of totally mixed strategy σ^n. To obtain a sequential equilibrium, one must construct beliefs β such that (σ, β) is consistent and sequentially rational. Because σ^n are totally mixed, associated beliefs β^n are uniquely defined by Bayes's rule. It suffice to take the limit β of a convergent subsequence β^n.

Lemma 8. *Let σ be a perfect/quasi-perfect equilibrium. Then, there is a belief system β such that (σ, β) is a sequential equilibrium.*

Theorem 13. *Every extensive form game contains a sequential equilibrium.*

Lemma 9. *Let* (σ, β) *be a sequential equilibrium. Then,* σ *is a subgame perfect equilibrium.*

Lemma 10. *A sequential equilibrium is almost perfect Bayesian.*

Note: Sequential equilibrium is very close to (and is implied by) trembling hand perfect in the extensive form. Roughly speaking, sequential equilibrium requires behavior to be sequentially rational against a limit assessment, while trem- bling hand perfection in the extensive form requires the behavior to be sequentially rational along the sequence as well.

4.2.2 Comment

The notion of sequential rationality requires players, at every information set of the game, to assign positive probability only to those opponents' strategies which are a sequential best response at this information set. In particular, they should do so at information sets which can only be reached if one of the opponents, say player i, has played a strategy that is not a sequential best response. Even in this case, $i's$ opponents should believe that player i will choose a sequential best response in the future. Similar to backward induction and subgame perfect equilibrium, the reasonableness of this property may be questioned in some examples.

4.3 Forward Induction

There are many forward induction criteria: restrict the way in which players should revise their conjectures during the game. Although there does not exist a unique definition of forward induction in the literature, its main idea can be described as follows. Suppose that a player's initial conjecture about the opponents' behavior has been contradicted by the play of the game. In this case, this player is required to form an alternative conjecture compatible with the current situation in the game. In other words, the player should study all possible scenarios that could have led to this situation, and form some subjective belief about these scenarios. Roughly speaking, a forward induction criterion requires the player to make a distinction between "more plausible" and "less plausible" scenarios, and to assign positive prob- ability only to the "more plausible" scenarios. The idea of forward induction is particularly successful in eliminating "implausible" sequential equilibria in signaling games.

4.4 Iterated Weak Dominance

Skip.

4.5 Stable Sets of Equilibria

Skip.

4.6 Forward Induction Equilibrium

4.6.1 Signaling Game

4.6.2 Intuitive criterion (Cho and Kreps, 1987)

One sentence summary: the principle of discarding player i actions that, from player $i's$ viewpoint, are dominated by his equilibrium utility.